1.2

Rules for Finding Limits

This section presents theorems for calculating limits. The first three let us build on the results of Example 8 in the preceding section to find limits of polynomials, rational functions, and powers. The fourth prepares for calculations later in the text.

Limits of Powers and Algebraic Combinations

Theorem 1

Properties of Limits

The following rules hold if $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ (L and M real numbers).

1. *Sum Rule:* $\lim_{x\to c} [f(x) + g(x)] = L + M$
2. *Difference Rule:* $\lim_{x\to c} [f(x) - g(x)] = L - M$
3. *Product Rule:* $\lim_{x\to c} f(x) \cdot g(x) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to c} kf(x) = kL$ (any number k)
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$
6. *Power Rule:* If m and n are integers, then

$$\lim_{x\to c} [f(x)]^{m/n} = L^{m/n},$$

provided $L^{m/n}$ is a real number.

In words, the formulas in Theorem 1 say:

1. The limit of the sum of two functions is the sum of their limits.
2. The limit of the difference of two functions is the difference of their limits.
3. The limit of the product of two functions is the product of their limits.
4. The limit of a constant times a function is that constant times the limit of the function.
5. The limit of the quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.
6. The limit of any rational power of a function is that power of the limit of the function, provided the latter is a real number.

We will prove the Sum Rule in Section 1.3. Rules 2–5 are proved in Appendix 2. Rule 6 is proved in more advanced texts.

EXAMPLE 1 Find $\lim_{x\to c} \dfrac{x^3 + 4x^2 - 3}{x^2 + 5}$.

Solution Starting with the limits $\lim_{x\to c} x = c$ and $\lim_{x\to c} k = k$ from Section 1.1, Example 8, and combining them using various parts of Theorem 1, we obtain:

a) $\lim_{x \to c} x^2 = \left(\lim_{x \to c} x\right)\left(\lim_{x \to c} x\right) = c \cdot c = c^2$ — Product or Power

b) $\lim_{x \to c} (x^2 + 5) = \lim_{x \to c} x^2 + \lim_{x \to c} 5 = c^2 + 5$ — Sum and (a)

c) $\lim_{x \to c} 4x^2 = 4 \lim_{x \to c} x^2 = 4c^2$ — Constant Multiple and (a)

d) $\lim_{x \to c} (4x^2 - 3) = \lim_{x \to c} 4x^2 - \lim_{x \to c} 3 = 4c^2 - 3$ — Difference and (c)

e) $\lim_{x \to c} x^3 = \left(\lim_{x \to c} x^2\right)\left(\lim_{x \to c} x\right) = c^2 \cdot c = c^3$ — Product and (a), or Power

f) $\lim_{x \to c} (x^3 + 4x - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} (4x^2 - 3)$ — Sum

$= c^3 + 4c^2 - 3$ — (d) and (e)

g) $\lim_{x \to c} \dfrac{x^3 + 4x^2 - 3}{x^2 + 5} = \dfrac{\lim_{x \to c} (x^3 + 4x^2 - 3)}{\lim_{x \to c} (x^2 + 5)}$ — Quotient

$= \dfrac{c^3 + 4c^2 - 3}{c^2 + 5}$ — (f) and (b)

EXAMPLE 2 Find $\lim_{x \to -2} \sqrt{4x^2 - 3}$.

Solution

$$\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{4(-2)^2 - 3} \qquad \text{Example 1(d) and Power Rule with } n = 1/2$$

$$= \sqrt{16 - 3}$$

$$= \sqrt{13}$$

□

Two consequences of Theorem 1 further simplify the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, merely substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point c *at which the denominator is not zero*, substitute c for x in the formula for the function.

Theorem 2

Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

Theorem 3

Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Exercises 1.2

Limit Calculations

Find the limits in Exercises 1–16.

1. $\lim_{x\to -7} (2x+5)$
2. $\lim_{x\to 12} (10-3x)$
3. $\lim_{x\to 2} (-x^2+5x-2)$
4. $\lim_{x\to -2} (x^3-2x^2+4x+8)$
5. $\lim_{t\to 6} 8(t-5)(t-7)$
6. $\lim_{s\to 2/3} 3s(2s-1)$
7. $\lim_{x\to 2} \dfrac{x+3}{x+6}$
8. $\lim_{x\to 5} \dfrac{4}{x-7}$
9. $\lim_{y\to -5} \dfrac{y^2}{5-y}$
10. $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$
11. $\lim_{x\to -1} 3(2x-1)^2$
12. $\lim_{x\to -4} (x+3)^{1984}$
13. $\lim_{y\to -3} (5-y)^{4/3}$
14. $\lim_{z\to 0} (2z-8)^{1/3}$
15. $\lim_{h\to 0} \dfrac{3}{\sqrt{3h+1}+1}$
16. $\lim_{h\to 0} \dfrac{5}{\sqrt{5h+4}+2}$

Find the limits in Exercises 17–30. Rationalization.

17. $\lim_{x\to 5} \dfrac{x-5}{x^2-25}$
18. $\lim_{x\to -3} \dfrac{x+3}{x^2+4x+3}$
19. $\lim_{x\to -5} \dfrac{x^2+3x-10}{x+5}$
20. $\lim_{x\to 2} \dfrac{x^2-7x+10}{x-2}$
21. $\lim_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$
22. $\lim_{t\to -1} \dfrac{t^2+3t+2}{t^2-t-2}$
23. $\lim_{x\to -2} \dfrac{-2x-4}{x^3+2x^2}$ Common bahir
24. $\lim_{y\to 0} \dfrac{5y^3+8y^2}{3y^4-16y^2}$
25. $\lim_{u\to 1} \dfrac{u^4-1}{u^3-1}$
26. $\lim_{v\to 2} \dfrac{v^3-8}{v^4-16}$
27. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$
28. $\lim_{x\to 4} \dfrac{4x-x^2}{2-\sqrt{x}}$
29. $\lim_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$
30. $\lim_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

Using Limit Rules

31. Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0}\frac{2f(x)-g(x)}{(f(x)+7)^{2/3}} = \frac{\lim_{x\to 0}(2f(x)-g(x))}{\lim_{x\to 0}(f(x)+7)^{2/3}} \qquad \text{(a)}$$

$$= \frac{\lim_{x\to 0} 2f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0}(f(x)+7)\right)^{2/3}} \qquad \text{(b)}$$

$$= \frac{2\lim_{x\to 0} f(x) - \lim_{x\to 0} g(x)}{\left(\lim_{x\to 0} f(x) + \lim_{x\to 0} 7\right)^{2/3}} \qquad \text{(c)}$$

$$= \frac{(2)(1)-(-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

32. Let $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$, and $\lim_{x\to 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 1}\frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim_{x\to 1}\sqrt{5h(x)}}{\lim_{x\to 1}(p(x)(4-r(x)))} \qquad \text{(a)}$$

$$= \frac{\sqrt{\lim_{x\to 1} 5h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1}(4-r(x))\right)} \qquad \text{(b)}$$

$$= \frac{\sqrt{5\lim_{x\to 1} h(x)}}{\left(\lim_{x\to 1} p(x)\right)\left(\lim_{x\to 1} 4 - \lim_{x\to 1} r(x)\right)} \qquad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4-2)} = \frac{5}{2}$$

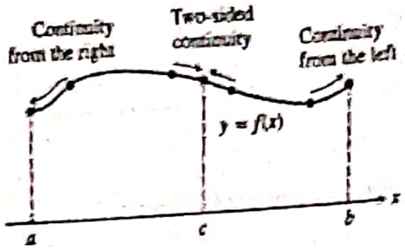
33. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find
    a) $\lim_{x\to c} f(x)g(x)$
    b) $\lim_{x\to c} 2f(x)g(x)$
    c) $\lim_{x\to c} (f(x)+3g(x))$
    d) $\lim_{x\to c} \dfrac{f(x)}{f(x)-g(x)}$

34. Suppose $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find
    a) $\lim_{x\to 4} (g(x)+3)$
    b) $\lim_{x\to 4} xf(x)$
    c) $\lim_{x\to 4} (g(x))^2$
    d) $\lim_{x\to 4} \dfrac{g(x)}{f(x)-1}$

35. Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find
    a) $\lim_{x\to b} (f(x)+g(x))$
    b) $\lim_{x\to b} f(x)\cdot g(x)$
    c) $\lim_{x\to b} 4g(x)$
    d) $\lim_{x\to b} f(x)/g(x)$

36. Suppose that $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, and $\lim_{x\to -2} s(x) = -3$. Find
    a) $\lim_{x\to -2} (p(x)+r(x)+s(x))$
    b) $\lim_{x\to -2} p(x)\cdot r(x)\cdot s(x)$
    c) $\lim_{x\to -2} (-4p(x)+5r(x))/s(x)$

Continuity at endpoints is defined by taking one-sided limits.



1.37 Continuity at points a, b, and c.

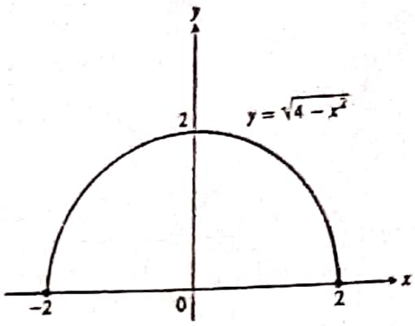
Definition

A function f is **continuous at a left endpoint** $x = a$ of its domain if

$$\lim_{x \to a^+} f(x) = f(a)$$

and **continuous at a right endpoint** $x = b$ of its domain if

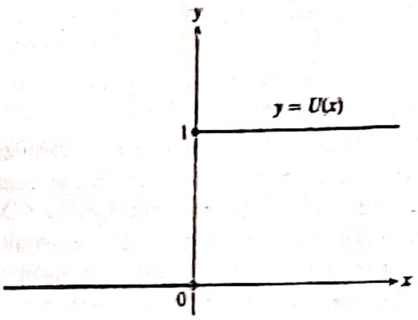
$$\lim_{x \to b^-} f(x) = f(b).$$



1.38 Continuous at every domain point.

In general, a function f is **right-continuous (continuous from the right)** at a point $x = c$ in its domain if $\lim_{x \to c^+} f(x) = f(c)$. It is **left-continuous (continuous from the left)** at c if $\lim_{x \to c^-} f(x) = f(c)$. Thus, a function is continuous at a left endpoint a of its domain if it is right-continuous at a and continuous at a right endpoint b of its domain if it is left-continuous at b. A function is continuous at an interior point c of its domain if and only if it is both right-continuous and left-continuous at c (Fig. 1.37).

EXAMPLE 1 The function $f(x) = \sqrt{4 - x^2}$ is continuous at every point of its domain, $[-2, 2]$ (Fig. 1.38). This includes $x = -2$, where f is right-continuous, and $x = 2$, where f is left-continuous. ❑



1.39 Right-continuous at the origin.

EXAMPLE 2 The unit step function $U(x)$, graphed in Fig. 1.39, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. ❑

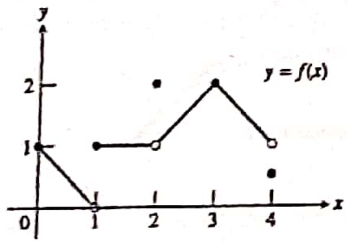
We summarize continuity at a point in the form of a test.

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

For one-sided continuity and continuity at an endpoint, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.



1.40 This function, defined on the closed interval [0, 4], is discontinuous at $x = 1, 2,$ and 4. It is continuous at all other points of its domain.

EXAMPLE 3 Consider the function $y = f(x)$ in Fig. 1.40, whose domain is the closed interval [0, 4]. Discuss the continuity of f at $x = 0, 1, 2, 3,$ and 4.

11. Show that the function $f :]0, 1] \to R$ defined by

$$f(x) = \frac{1}{x}$$

is continuous on $]0, 1]$. Is $f(x)$ bounded on this interval? Explain.

12. Let $f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Is f continuous at $x = 0$?

13. Let $f(x) = \begin{cases} (x-a)\sin\left(\frac{1}{x-a}\right) & \text{if } x \neq a \\ 0 & \text{if } x = a \end{cases}$

Discuss the continuity of f at $x = a$

14. Let $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Show that f is continuous at $x = 0$

15. Let $f(x) = \begin{cases} x^2\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = 0$

16. Let $f(x) = \begin{cases} x\sin\left(\frac{\lfloor x \rfloor}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = 0$

17. Find c such that the function

$$f(x) = \begin{cases} \dfrac{1-\sqrt{x}}{x-1} & \text{if } 0 \leq x < 1 \\ c & \text{if } x = 1 \end{cases}$$

is continuous for all $x \in [0, 1]$.

In Problems 18 – 20, find the points of discontinuity of the given functions.

18. $f(x) = \begin{cases} x+4 & \text{if } -6 \leq x < -2 \\ x & \text{if } -2 \leq x < 2 \\ x-4 & \text{if } 2 \leq x < 6 \end{cases}$

19. $g(x) = \begin{cases} x^3 & \text{if } x < 1 \\ -4-x^2 & \text{if } 1 \leq x \leq 10 \\ 6x^2+46 & \text{if } x > 10 \end{cases}$